VARIATIONAL SOLUTION OF THIRD BOUNDARY-VALUE PROBLEM OF HEAT CONDUCTION

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It is shown that the Ainola variational principle can be used to solve the third boundary-value problem of heat conduction.

L. Ya. Ainola has given a variational principle [1] for solution of the first and second heat-conduction boundary-value problems. Here we shall show that this principle can be used to solve the third linear one-dimensional boundary-value problem which is formulated as

$$\frac{1}{x^{m-1}} \frac{\partial}{\partial x} \left[\lambda(x) x^{m-1} \frac{\partial T(x,\tau)}{\partial x} \right] = c\rho(x) \frac{\partial T(x,\tau)}{\partial \tau} - q_v(x,\tau), \quad a < x < b, \quad \tau > 0 \tag{1}$$

(m = 1, 2, 3 for a plate, cylinder, and sphere, respectively) under the initial condition

$$T(x, 0) = \varphi(x), \quad a < x < b \tag{2}$$

and the boundary condition

$$\lambda(a) \frac{\partial T(a,\tau)}{\partial x} + \alpha_1 \left[T_{a_1}(\tau) - T(a,\tau) \right] = 0, \quad \tau > 0, \tag{3}$$

$$-\lambda(b) \frac{\partial T(b,\tau)}{\partial x} + \alpha_2 \left[T_{a_2}(\tau) - T(b,\tau) \right] = 0, \quad \tau > 0. \tag{4}$$

Thus we consider the variation of the temperature T only in the x direction and during the time τ when the thermal conductivity λ and the bulk specific heat c_p depend on x and the body contains a bulk heat-evolving source of power $q_v(x, \tau)$. The convective heat-transfer coefficients α_1 and α_2 are assumed to be constant, and the initial temperature distribution $T(x, 0) = \varphi(x)$ and the variation of the temperatures in the ambient medium $T_{a_1}(\tau)$ and $T_{a_2}(\tau)$ are assumed to be continuous functions that together with their first derivatives satisfy the congruence conditions

$$\lambda(a) \varphi'(a) + \alpha_1 \left[T_{a_1}(0) - \varphi(a) \right] = 0, \tag{5}$$

$$-\lambda(b) \varphi'(b) + \alpha_2 \left[T_{a_2}(0) - \varphi(b) \right] = 0. \tag{6}$$

The lack of such congruence in the formulation of the problem is easily remedied by the computational method described below.

We introduce the new unknown function $u(x, \tau)$ such that

$$T(x,\tau) = u(x,\tau) + \varphi(x) + \left[T_{a_1}(\tau) - T_{a_1}(0) \right] \frac{(b-x)^2 (2x+b-3a)}{(b-a)^3} + \left[T_{a_2}(\tau) - T_{a_2}(0) \right] \frac{(x-a)^2 (3b-a-2x)}{(b-a)^3}, \tag{7}$$

and then reduce problem (1)-(4) to homogeneous initial and boundary conditions:

$$\frac{1}{x^{m-1}} \frac{\partial}{\partial x} \left[\lambda(x) x^{m-1} \frac{\partial u(x,\tau)}{\partial x} \right] - c\rho(x) \frac{\partial u(x,\tau)}{\partial \tau} - f(x,\tau) = 0, \tag{1'}$$

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$$a<x<b,\ \tau>0,\ m=1,\ 2,\ 3,$$

$$u(x,\ 0)=0,\ a<x<b, \tag{2'}$$

$$-\lambda(a)\frac{\partial u(a,\ \tau)}{\partial x}+\alpha_1 u(a,\ \tau)=0,\quad \tau>0, \tag{3'}$$

$$\lambda(b)\frac{\partial u(b,\ \tau)}{\partial x}+\alpha_2 u(b,\ \tau)=0,\quad \tau>0, \tag{4'}$$

where

$$f(x,\ \tau)=\frac{c\rho(x)}{(b-a)^3}\left[T'_{a_1}(\tau)(b-x)^2(2x+b-3a)+T'_{a_2}(\tau)(x-a)^2\times\right.$$
$$\times(3b-a-2x)]-q_v(x,\ \tau)-\frac{1}{x^{m-1}}\frac{\partial}{\partial x}\{\{x^{m-1}\lambda(x)\{\varphi'(x)+$$
$$+\frac{6(b-a)(x-a)}{(b-a)^3}[T_{a_2}(\tau)-T_{a_2}(0)-T_{a_1}(\tau)+T_{a_1}(0)]\}\}. \tag{8}$$

In accordance with [1] we set up the functional in u for the finite temporal [0, t] and spatial [a, b] intervals:

$$J(u)=\int_0^t\int_a^b\left\{\frac{1}{x^{m-1}}\frac{\partial}{\partial x}\left[\lambda(x)x^{m-1}\frac{\partial u(x,\ \tau)}{\partial x}\right]-c\rho(x)\frac{\partial u(x,\ \tau)}{\partial\tau}-\right.$$
$$\left.-2f(x,\ \tau)\right\}x^{m-1}u(x,\ t-\tau)\,dxd\tau. \tag{9}$$

Integrating the first term on the right side of (9) by parts with respect to x and using (3') and (4') we obtain

$$J(u)=-\int_0^t\left\{\int_a^b\left[\lambda(x)\frac{\partial u(x,\ \tau)}{\partial x}\frac{\partial u(x,\ t-\tau)}{\partial x}+\right.\right.$$
$$+c\rho(x)\frac{\partial u(x,\ \tau)}{\partial\tau}u(x,\ t-\tau)+2f(x,\ \tau)u(x,\ t-\tau)\Big]x^{m-1}dx+$$
$$+2\left[\alpha_2 b^{m-1}u(b,\ \tau)u(b,\ t-\tau)+\alpha_1 a^{m-1}u(a,\ \tau)u(a,\ t-\tau)\right]\Big\}d\tau. \tag{9'}$$

We represent the variation of u(x, τ) by $\zeta(x,\ \tau)$ and make use of the convolution-symmetry property to obtain the following form of variation for the given functional:

$$\delta J(u)=-\int_0^t\left\{\left\{\int_a^b 2\lambda(x)\frac{\partial u(x,\ \tau)}{\partial x}\frac{\partial\zeta(x,\ t-\tau)}{\partial x}+c\rho(x)\times\right.\right.$$
$$\times\left[u(x,\ \tau)\frac{\partial\zeta(x,\ t-\tau)}{\partial\tau}+\frac{\partial u(x,\ \tau)}{\partial\tau}\zeta(x,\ t-\tau)\right]+$$
$$+2f(x,\ \tau)\zeta(x,\ t-\tau)\Big\}x^{m-1}dx+2\left[\alpha_2 b^{m-1}u(b,\ \tau)\zeta(b,\ t-\tau)+\right.$$
$$\left.+\alpha_1 a^{m-1}u(a,\ \tau)\zeta(a,\ t-\tau)\right]\Big\}\Big\}d\tau. \tag{10'}$$

Next, we integrate the first term in (10) by parts with respect to x:

$$\int_a^b\lambda(x)x^{m-1}\frac{\partial u(x,\ \tau)}{\partial x}\frac{\partial\zeta(x,\ t-\tau)}{\partial x}dx=\lambda(b)b^{m-1}\frac{\partial u(b,\ \tau)}{\partial x}\zeta(b,\ t-\tau)-$$
$$-\lambda(a)a^{m-1}\frac{\partial u(a,\ \tau)}{\partial x}\zeta(a,\ t-\tau)-\int_a^b\frac{\partial}{\partial x}\left[\lambda(x)x^{m-1}\frac{\partial u(x,\ \tau)}{\partial x}\right]\times$$
$$\times\zeta(x,\ t-\tau)\,dx$$

and on the basis of the convolution symmetry also establish that

$$\int_0^t u(x,\ \tau)\frac{\partial\zeta(x,\ t-\tau)}{\partial\tau}d\tau=\int_0^t\frac{\partial u(x,\ \tau)}{\partial\tau}\zeta(x,\ t-\tau)\,d\tau.$$

Then (10) takes the form

$$\delta J=2\int_0^t\left\{\left\{\int_a^b\left\{\frac{1}{x^{m-1}}\left[\frac{\partial}{\partial x}\lambda(x)x^{m-1}\frac{\partial u(x,\ \tau)}{\partial x}\right]-c\rho(x)\frac{\partial u(x,\ \tau)}{\partial\tau}-\right.\right.\right.$$
$$-f(x,\ \tau)\Big\}x^{m-1}\zeta(x,\ t-\tau)\,dx+b^{m-1}\left[\lambda(b)\frac{\partial u(b,\ \tau)}{\partial x}+\right.$$

$$+\alpha_2 u(b, \tau)\Big] \zeta(b, t-\tau) + a^{m-1}\Big[-\lambda(a)\frac{\partial u(a, \tau)}{\partial x} + \alpha_1 u(a, \tau)\Big] \zeta(a, t-\tau)\}\} d\tau. \tag{11}$$

Considering (11), we conclude that the function $u(x, \tau)$ satisfying the condition $\delta J = 0$ for $a < x < b$, $0 < \tau < t$ is a solution of the problem (1')-(4').

Following L. V. Kantorovich [2], we seek a solution of (1')-(4') in the first approximation in the form

$$u(x, \tau) = g(x)\psi(\tau), \tag{12}$$

where $g(x)$ is a known function of the coordinates that satisfies conditions (3')-(4'), while $\psi(\tau)$ is the desired unknown time function satisfying the condition $\psi(0) = 0$.

It is not difficult to show that for our problem in which $u(x, \tau)$ is determined in the form (12), the Euler equation (the condition for stationarity of the functional) will be

$$A\psi(t-\tau) - B\psi'(t-\tau) - C(t-\tau) = 0,$$

where

$$A = \int_a^b [\lambda(x) x^{m-1} g'(x)]' g(x)\, dx;$$

$$B = \int_a^b c\rho(x) x^{m-1} g^2(x)\, dx;$$

$$C(\tau) = \int_a^b f(x, \tau) g(x) x^{m-1}\, dx.$$

Thus in first approximation the solution of (1)-(4) will have the form

$$T(x, \tau) = g(x)\psi(\tau) + \varphi(x) + [T_{a_1}(\tau) - T_{a_1}(0)] \times \frac{(b-x)^2(2x+b-3a)}{(b-a)^3} + [T_{a_2}(\tau) - T_{a_2}(0)]\frac{(x-a)^2(3b-a-2x)}{(b-a)^3}.$$

The succeeding approximations are found in like manner.

To illustrate the application of the above, let us solve the one-dimensional symmetric problem of heat conduction when the initial temperature T_0 of the body and the ambient temperature T_a are constant for a body with constant thermophysical properties λ, $c\rho$ having no source (sink) $q_V(x, \tau)$. Here we assume initially that the temperature of the medium varies from T_0 to T_a over a small time interval $0 \leq \tau \leq \tau_0$, only then ($\tau_0 \leq \tau$) becoming constant:

$$T_a(\tau) = \begin{cases} \dfrac{T_0 - T_a}{\tau_0^2}(\tau - \tau_0)^2 + T_a, & 0 \leqslant \tau \leqslant \tau_0, \\ T_a, & \tau_0 \leqslant \tau. \end{cases}$$

Thus the congruence conditions (5), (6) are provided. Next we introduce the dimensionless coordinate $\eta = x/l_0$, the dimensionless time $\bar{\tau} = a\tau/l_0^2 (a = \lambda/c\rho)$, and the dimensionless intensity of heat transfer (the Biot number) $\mathrm{Bi} = \alpha l_0/\lambda$ and assume that $T(\eta, \bar{\tau}) = u(\eta, \bar{\tau}) + T_a(\bar{\tau})$. Then our problem takes the form

$$\frac{1}{\eta^{m-1}}\frac{\partial}{\partial \eta}\left[\eta^{m-1}\frac{\partial u(\eta, \bar{\tau})}{\partial \eta}\right] = \frac{\partial u(\eta, \bar{\tau})}{\partial \bar{\tau}} + T_a'(\bar{\tau}), \tag{13}$$

$$-1 < \eta < 1, \quad \bar{\tau} > 0, \quad m = 1, 2, 3,$$

$$u(\eta, 0) = 0, \quad -1 < \eta < 1, \tag{14}$$

$$-\frac{\partial u(-1, \bar{\tau})}{\partial \eta} + \mathrm{Bi}\, u(-1, \bar{\tau}) = 0, \ \bar{\tau} > 0, \tag{15}$$

$$\frac{\partial u(1, \bar{\tau})}{\partial \eta} + \mathrm{Bi}\, u(1, \bar{\tau}) = 0, \quad \bar{\tau} > 0. \tag{16}$$

It can be shown that solution of the given problem when

$$u(\eta, \bar{\tau}) = \left(\frac{2}{\mathrm{Bi}} + 1 - \eta^2\right)\psi(\bar{\tau})$$

is chosen as the first approximation is equivalent to solution of the following Euler equation:

$$\psi'(\bar{\tau}) + \mu_1 \psi(\bar{\tau}) = -\frac{\mu_1}{2m} T_a'(\bar{\tau}), \tag{17}$$

where

$$\mu_1 = \frac{m(m+4)\,\mathrm{Bi}\,(\mathrm{Bi}+m+2)}{2\mathrm{Bi}+2(m+4)\,\mathrm{Bi}+(m+2)(m+4)};$$

$$T_a'(\bar{\tau}) = \begin{cases} \dfrac{2(T_0 - T_a)}{\bar{\tau}_0^2}(\bar{\tau} - \bar{\tau}_0), & \bar{\tau} < \bar{\tau}_0, \\ 0, & \bar{\tau} > \bar{\tau}_0. \end{cases}$$

Solving (17), we obtain $\psi(\bar{\tau})$. We shall not give the computational relationships separately for $0 \le \bar{\tau} \le \bar{\tau}_0$ and $\bar{\tau}_0 \le \bar{\tau}$ but shall just give the formula for Θ when $\bar{\tau}_0 \to 0$ (to permit comparison with the classical solutions, and owing to space limitations):

$$\Theta(\eta, \bar{\tau}) = \frac{T(\eta, \bar{\tau}) - T_a}{T_0 - T_a} = \frac{\mu_1}{2m}\left(\frac{2}{\mathrm{Bi}} + 1 - \eta^2\right)\exp(-\mu_1 \bar{\tau}). \tag{18}$$

Similarly, taking

$$u(\eta, \bar{\tau}) = \left(\frac{2}{\mathrm{Bi}} + 1 - \eta^2\right)\psi_1(\bar{\tau}) + \left(\frac{2}{\mathrm{Bi}+2} + 1 - \eta^2\right)\eta^2 \psi_2(\bar{\tau}),$$

for the second approximation, we obtain

$$\psi_1(\bar{\tau}) = \frac{m+6}{8}\,\frac{\mathrm{Bi}}{\mathrm{Bi}+2}\,\frac{D}{F}\,\frac{T_a - T_0}{\mu_2 - \mu_1}\left[(\mu_1 - \beta)\exp(-\mu_1 \bar{\tau}) - (\mu_2 - \beta)\exp(-\mu_2 \bar{\tau})\right], \tag{19}$$

$$\psi_2(\bar{\tau}) = \frac{(m+6)(m+8)}{8}\,\frac{\mathrm{Bi}\,[\mathrm{Bi}+2(m+4)]}{F}\,\frac{T_a - T_0}{\mu_2 - \mu_1}\left[\mu_1 \exp(-\mu_1 \bar{\tau}) - \mu_2 \exp(-\mu_2 \bar{\tau})\right]. \tag{20}$$

after carrying out the corresponding steps for $\bar{\tau}_0 \to 0$. In the formulas for $\psi(\bar{\tau})$,

$$D = (4-m)\,\mathrm{Bi}^2 + 2(4-m)(m+6)\,\mathrm{Bi} + 8(5m+22),$$
$$F = 3\mathrm{Bi}^2 + 6(m+6)\,\mathrm{Bi} + (m+6)(5m+22),$$

$$\beta = \frac{4(m+2)(m+8)(\mathrm{Bi}+2)[\mathrm{Bi}+2(m+4)]}{D},$$

$$\left.\begin{matrix}\mu_1 \\ \mu_2\end{matrix}\right\} = \frac{(m+2)(m-6)}{F}\left\{2\mathrm{Bi}^2 + 4(m+5)\,\mathrm{Bi} + (m+4)(m+8) \mp \sqrt{[2\mathrm{Bi}^2 + 4(m+5)\,\mathrm{Bi} + (m+4)(m+8)]^2 - \frac{m(m+8)}{(m+2)(m+6)} \times \mathrm{Bi}\,[\mathrm{Bi}+2(m+4)]\,F}\right\}.$$

The quantities $\mu_1^{0.5}$ in the first and second approximations coincide exactly with the exact values of the first root of the characteristic equations of heat conduction tabulated in [3]. The second root of the characteristic equations given in [3] differs negligibly from $\mu_2^{0.5}$ (the discrepancy decreases from 6 to 0.1% as Bi decreases).

Analysis of the function $\Theta = \Theta(\eta, \tau)$ of the second approximation for monotonicity in the region $|\eta| \le 0.5$ and comparative digital-computer calculations show that the approximate values Θ agree well with the exact values Θ_T for all Bi when

$$\tau > \frac{1}{\mu_2 - \mu_1}\left[\frac{3}{2}\ln\frac{\mu_2}{\mu_1} + \ln\frac{(M+N)\mu_2 - M\beta}{(M+N)\mu_1 - M\beta}\right],$$

where

$$M = \frac{\dfrac{2}{\mathrm{Bi}} + 1 - \eta^2}{\mathrm{Bi}+2} D; \quad N = (m+8)(\mathrm{Bi}+2m+8)\left(\frac{2}{\mathrm{Bi}+2} + 1 - \eta^2\right)\eta^2.$$

In the region $|\eta| > 0.5$ a second-approximation error of less than 5% is observed beginning at $\bar{\tau} \ge 0.05$.

For the Bi = Bi($\bar{\tau}$) case, which is of practical importance in thermophysics, if we use the above approach and symmetrize the smooth function Bi($\bar{\tau}$) at the point $\bar{\tau}$ on the interval [0, $2\bar{\tau}$], as we did in [4], we obtain

$$u(\eta, \bar{\tau}) = -\left[\frac{2}{\mathrm{Bi}(\bar{\tau})} + 1 - \eta^2\right]\frac{m+4}{2}\exp\left[-\frac{m(m+4)}{2}\int_0^{\bar{\tau}} \nu(z)\,dz\right]\times$$

$$\times\int_0^{\bar{\tau}} \nu(\gamma)\,T_a'(\gamma)\exp\left[\frac{m+4}{2}\int_0^{\gamma}\nu(z)\,dz\right]d\gamma,$$

in the first approximation, where

$$\nu(z) = \frac{\mathrm{Bi}(z)\,[\mathrm{Bi}(z) + m + 2]}{2\mathrm{Bi}^2(z) + 2(m+4)\,\mathrm{Bi}(z) + (m+2)(m+4)}.$$

Problems of this sort have so far been solved only for certain Bi = Bi($\bar{\tau}$) dependences [5].

To conclude we note that a separate paper will be devoted to analytic evaluation of the order of convergence and of the solution error for the approximate method used.

NOTATION

$\Theta = [T(x, \tau) - T_0]/[T_a - T_0]$ and $T(x, \tau)$, T_0 and T_a	are the dimensionless and dimensioned running temperatures, the initial temperature of the body, and the ambient temperature;
x and $\eta = x/l_0$, l_0	are the dimensioned and dimensionless coordinates of a point in the body and the characteristic length of the body;
τ and $\bar{\tau} = a\tau/l_0^2$	are the dimensioned and dimensionless time;
$a = \lambda/c_p$ and λ, c_p	are the thermal diffusivity and thermal conductivity, and the bulk specific heat;
$q_v(x, \tau)$	is the power of the bulk heat-evolution source;
α and $\mathrm{Bi} = \alpha l_0/\lambda$	are the heat-transfer coefficient and the Biot number.

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